EXAM SETS & NUMBERS (PART 1: SETS), January 30th, 2025, 8:30-10:30pm, Aletta Jacobshal 04 T13-X9.

Write your name on every sheet of paper that you intend to hand in. Please provide **complete** *arguments for each of your answers. This part of the exam consists of 2 questions. You can score up to 9 points for each question, and you obtain 2 points for free. In this way you will score in total between 2 and 20 points.*

(1) (a) [2 points] Give an example of a set X that has elements x and y such that $x \in y$.

(b) [2 points] We say a set $\mathcal{A}$ is nice iff it has the following two properties: 1) all its elements are sets and 2) if $A \in \mathcal{A}$ and $B \subseteq A$ then $B \in \mathcal{A}$. Prove that if $\mathcal{A}$ and $\mathcal{B}$ are nice then so is $\mathcal{A} \cap \mathcal{B}$.

(c) [2+1 points] Define a relation $\sim$ on $2^{\mathbb{N}}$ as follows. $A \sim B$ iff $(A \cap B)^c$ is finite. Prove that this relation is symmetric and transitive. For $X \subseteq \mathbb{N}$ define $X' = \{Y \in 2^{\mathbb{N}} : Y \sim X\}$. Is it true that $X \in X'$ for all infinite sets X?

(d) [2 points] **Point out why the following is incorrect:** If X and Y are infinite sets then there must be an invertible function $f : X \to Y$. Proof: By definition there are bijections $\alpha : X \to \mathbb{N}$ and $\beta : Y \to \mathbb{N}$ so we can take $f = \beta^{-1} \circ \alpha$. Since the composition of two bijections is again a bijection we conclude that f is indeed invertible.

(2) (a) [2 points] For all $n \in \mathbb{N}$ we take a set X_n with cardinality n. Prove by induction that $\#\bigcup_{j=0}^{n} X_j \leq n^2$ is true for all natural numbers n.

(b) [2 points] On the set $F = [2]^{[2]}$ define the following equivalence relation: $f \sim g$ iff there are invertible functions $h, k \in F$ such that $f = h \circ g \circ k$. How many equivalence classes are there?

(c) [1+2 points] Give the domain, co-domain and range of the function id_X where X is a finite set. Do the same for function $f : 2^X \to \mathbb{Q}$ defined by $f(A) = \#A$.

(d) [2 points] Define a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ by $f(a, b) = a - b$. Give an explicit description of the set $f^{-1}(O)$, where O is the set of all odd numbers.

If you are only retaking the sets part, this is the side you need to complete. Otherwise, please turn over for part 2 on numbers and do that part on a DIFFERENT piece of paper.